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1993 J. Phys. A: Math. Gen. 26 L375

(http://iopscience.iop.org/0305-4470/26/7/005)

J. Phys. A: Math. Gen. 26 (1993) L375–L378. Printed in the UK

LETTER TO THE EDITOR

A matrix spectral problem in one complex space dimension

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Received 5 January 1993

Abstract. The inverse *spectral* method for an $N \times N$ spectral problem is studied via the $\bar{\partial}$-problem for a one-dimensional complex space. The complex mKdV equations are explicitly solved as an example.

Spectral problems in one (real) space dimension are now classical, see e.g. [1–4]. The main structure has been solidly founded and high-order poles [5, 6] were introduced algebraically to ZS–AKNS and $N \times N$ spectral equations. This past decade has also seen some intensive research in the inverse scattering methods (ISMs) for multi-dimensions [7–15]. A number of physically or mathematically important nonlinear evolution equations (NEEs) are solved this way. Multi-dimensional inverse problems of a different kind are also well studied for such as the Schrödinger equation [16]. Up to now, the multi-dimensional scattering operators that are related to integrable NEEs such as KP and DS equations [7–10] essentially do not contain any spectral parameters in the sense that any such parameters can be transformed away, while those operators that contain genuine spectral parameters [16] are generally not known to be associated with any integrable NEEs through Lax pairs. In this letter, however, we shall consider the spectral equation in a complex space $z = x + \mathrm{i}y$

$$\psi_z = \lambda(k)\psi + u(z)\psi \tag{1}$$

where $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, λ=diag$(\lambda_1, \ldots, \lambda_N)$ and u are $N \times N$ matrices, $\lambda_l \not\equiv \lambda_m$ $\forall l \neq m$, $u \to 0$ for $|z| \to \infty$, and $\lambda(k)$ is analytic in the complex plane $\mathbb{C}$. Our new spectral problem (1) is not only a two (real) space dimensional *spectral* equation because in general $\lambda(k)$ cannot be rescaled away, but is also associated with 2+1-dimensional NEEs through Lax pairs. Hence by solving (1), we will provide an ISM to solve various associated NEEs such as the complex mKdV equations

$$q_t \pm 6q^2 q_z + q_{zzz} = 0. \tag{2}$$

In fact ISM for (1) is a typical $\bar{\partial}$-problem first introduced to ISM in 1+1 dimensions by Beals and Coifman [4] and then fully developed to solve multi-dimensional problems [7–15].

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We now proceed to solve (1) and thus solve a new class of integrable 2+1-dimensional NEEs related to it. We shall denote the complex conjugate by an overline. Let $\psi = \phi(z,k)\exp\left[\mathrm{i}(\lambda z + \overline{\lambda z})\right]$, we rewrite (1) as

$$\partial_{\bar z}\left[\exp\left\{-\mathrm{i}(\lambda(k)z + \overline{\lambda(k)z})\right\}\psi\right] = \exp\left\{-\mathrm{i}(\lambda(k)z + \overline{\lambda(k)z})\right\}u\psi. \tag{3}$$

Applying to (3) the $\partial_{\bar z}^{-1}$ operator or the following generalized Cauchy formula

$$f(k) = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{D}} \frac{\partial_{\overline{k'}} f(k')}{k'-k}\,\mathrm{d}k' \wedge \mathrm{d}\overline{k'} + \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathcal{D}} \frac{f(k')}{k'-k}\,\mathrm{d}k' \tag{4}$$

with the normalization $\phi \to e$, the $N \times N$ unit matrix, for $|z| \to \infty$, we have

$$\begin{aligned}\phi(z,k) = e + \frac{1}{2\pi\mathrm{i}}\int_{\mathbf{R}} \frac{\mathrm{d}z' \wedge \mathrm{d}\overline{z'}}{z'-z}\exp\left[\mathrm{i}(\lambda(k)(z-z') + \overline{\lambda(k)(z-z')})\right]u(z')\phi(z',k)\\ \times\exp\left[-\mathrm{i}(\lambda(k)(z-z') + \overline{\lambda(k)(z-z')})\right] \equiv e + \hat{G}\phi\end{aligned} \tag{5}$$

where $\mathrm{d}z' \wedge \mathrm{d}\overline{z'} = -2\mathrm{i}\,\mathrm{d}z'_{\mathrm{R}}\mathrm{d}z'_{\mathrm{I}}$ and z'_{R} and z'_{I} are the real and imaginary part of z' respectively. We note that the integral equation (5) is very different from those related to the normal two space dimensional problems [7–15] in the sense that the later ones have *essentially* no explicit scattering parameters in the scattering equations. In fact, the Green's functions for the scattering equations in the latter cases are parameterized by somewhat 'enforced' scattering parameters, and these types of general cases were shown to allow proper definitions for the necessary inverse data [13] whenever the Green's functions are analytically well behaved. However, our equation (1) falls beyond such a scope and is therefore a new kind of its own.

Now by differentiating (5) we have

$$\partial_{\bar k}\phi(z,k) = \left[\mathrm{e}^{\mathrm{i}(\lambda z + \overline{\lambda z})}T(k)\mathrm{e}^{-\mathrm{i}(\lambda z + \overline{\lambda z})},\ \partial_{\bar k}\overline{\lambda(k)}\right] + \hat{G}(\partial_{\bar k}\phi) \tag{6}$$

where $[\cdot,\cdot]$ is the commutation bracket and

$$T(k) = \frac{1}{2\pi}\int_{\mathbf{R}} \mathrm{d}z \wedge \mathrm{d}\bar{z}\,\mathrm{e}^{-\mathrm{i}(\lambda z + \overline{\lambda z})}u(z)\phi(z,k)\mathrm{e}^{\mathrm{i}(\lambda z + \overline{\lambda z})}. \tag{7}$$

Notice that the following integral equation

$$\begin{aligned}\varphi_m = \exp\left[\mathrm{i}(\lambda_l - \lambda_m)z + \mathrm{i}\overline{(\lambda_l - \lambda_m)z}\right]\cdot e_l\\ + \frac{1}{2\pi\mathrm{i}}\int_{\mathbf{R}} \frac{\mathrm{d}z' \wedge \mathrm{d}\overline{z'}}{z'-z}\exp\left[\mathrm{i}(\lambda_m - \lambda)(z'-z) + \mathrm{i}\overline{(\lambda_m - \lambda)(z'-z)}\right]u\varphi_m\end{aligned} \tag{8}$$

can be transformed into

$$\begin{aligned}\exp\left[\mathrm{i}(\lambda_m - \lambda_l)z + \mathrm{i}\overline{(\lambda_m - \lambda_l)z}\right]\varphi_m\\ = e_l + \frac{1}{2\pi\mathrm{i}}\int_{\mathbf{R}} \frac{\mathrm{d}z' \wedge \mathrm{d}\overline{z'}}{z'-z}\exp\left[\mathrm{i}(\lambda_l - \lambda)(z'-z) + \mathrm{i}\overline{(\lambda_l - \lambda)(z'-z)}\right]u\varphi_m\\ \times\exp\left[\mathrm{i}(\lambda_m - \lambda_l)z' + \mathrm{i}\overline{(\lambda_m - \lambda_l)z'}\right]\end{aligned}$$

and is therefore solved by

$$\varphi_m(z,k) = \exp\left[\mathrm{i}(\lambda_l(k) - \lambda_m(k))z + \overline{\mathrm{i}(\lambda_l(k) - \lambda_m(k))z}\right]\phi_l(z,k). \tag{9}$$

Since (6) also takes the following more suitable form

$$\partial_{\bar{k}}\phi_m = \sum_{l=1}^{N} S_{lm}(k) e_l \exp\left[\mathrm{i}(\lambda_l - \lambda_m)z + \overline{\mathrm{i}(\lambda_l - \lambda_m)z}\right] + \hat{G}\phi_m$$

where the spectral data S_{lm} are defined by

$$S(k) = \left[T(k),\ \partial_{\bar{k}}\overline{\lambda(k)}\right] \tag{10}$$

equation (6) is solved via (8) and (9) by

$$\partial_{\bar{k}}\phi(z,k) = \phi(z,k)\exp\left[\mathrm{i}(\lambda(k)z + \overline{\lambda(k)z})\right] S(k) \exp\left[-\mathrm{i}(\lambda(k)z + \overline{\lambda(k)z})\right]. \tag{11}$$

We can see from (11) that the extension of the space variable x into $z = x + \mathrm{i}y$ has transformed the Riemann–Hilbert problem [3, 6] into a $\bar{\partial}$-problem (11). Discrete spectral data are not considered because the $\bar{\partial}$-problem region, the whole of $\mathbb{C}$ here, in general does not provide ground for poles.

For the inverse of the spectral data, we use (4) on ϕ with $\phi \to e$ ($|k| \to \infty$). The eigenfunction ϕ can then be solved from the following integral equation

$$\phi(z,k) = e + \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{C}} \frac{\mathrm{d}k' \wedge \mathrm{d}\overline{k'}}{(k'-k)} \times \exp\left[\mathrm{i}(\lambda(k')z + \overline{\lambda(k')z})\right]\phi(z,k')S(k')\exp\left[-\mathrm{i}(\lambda(k')z + \overline{\lambda(k')z})\right]. \tag{12}$$

As a simple application, we shall consider just the complex mKdV equations (2). Since the Lax pair for (2) is essentially that of one real space dimension [1]

$$\begin{aligned} \mathsf{L} &\equiv \partial_z - \begin{pmatrix} -\mathrm{i}k & q \\ r & \mathrm{i}k \end{pmatrix} \qquad r = \mp q \\ \mathsf{M} &\equiv \partial_t - \begin{pmatrix} -4\mathrm{i}k^3 - 2\mathrm{i}qrk & 4qk^2 + 2\mathrm{i}q_z k + 2q^2 r - q_{zz} \\ 4rk^2 - 2\mathrm{i}r_z k + 2qr^2 - r_{zz} & 4\mathrm{i}k^3 + 2\mathrm{i}qrk \end{pmatrix} \end{aligned} \tag{13}$$

equation (2) is also the compatibility condition for $\mathsf{L}\psi = 0$ and $\mathsf{M}\psi = -\psi\Omega(k)$ with $\Omega = \mathrm{diag}(-4\mathrm{i}k^3,\ 4\mathrm{i}k^3)$. By rewriting (11) as

$$\partial_{\bar{k}}\psi(z,k) = \psi(z,k)S(k) + \mathrm{i}\bar{z}\psi(z,k)\partial_{\bar{k}}\overline{\lambda(k)} \tag{14}$$

and applying the M operator in (13) to (14), we obtain $S(k,t) = \mathrm{e}^{\Omega(k)t}S(k,0)\mathrm{e}^{-\Omega(k)t}$. To complete the inverse of the spectral data, we only need to solve (12) and then, via the asymptotic of (1) for $k \to \infty$, the potential will finally be given by

$$\begin{pmatrix} & q \\ r & \end{pmatrix} = \left[\begin{pmatrix} -1 & \\ & 1 \end{pmatrix},\ \frac{1}{2\pi}\int_{\mathbb{C}} \mathrm{d}k \wedge \mathrm{d}\bar{k} \times \exp\left[\mathrm{i}(\lambda(k)z + \overline{\lambda(k)z})\right]\phi(z,k)S(k)\exp\left[-\mathrm{i}(\lambda(k)z + \overline{\lambda(k)z})\right]\right].$$

We have thus seen that the extension of a matrix spectral equation into a complex space dimension, apart from the interest as an ISM on its own, provides a *spectral* problem (in two real space dimensions) which can *also* be applied to solve NEEs. Finally we remark that NEEs solvable via ISMs are in general good candidates for complete integrability [1, 17].

The author would like to thank R K Bullough for the helpful discussions. This work was partially supported by a British SERC grant when the author was in UMIST, Manchester, UK.

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